

# The Damping of the Bose-Condensate Oscillations in a Trap at Zero Temperature

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We discuss an existence of the damping for the radial condensate oscillations in a cylindric trap at zero temperature. The damping is a result of the parametric resonance leading to energy transfer from the coherent condensate oscillations to the longitudinal sound waves within a finite frequency interval. The parametric resonance is due to the oscillations of the sound velocity. The triggering amplitudes at zero temperature are associated with the zero-point oscillations

## INTRODUCTION

The damping of the oscillations of a trapped Bose condensate isolated from environment is one of the most interesting problems in the physics of the Bose-Einstein condensation. So far, experimental [1-4] and theoretical [5-7] investigations have been reduced to studying the damping due to the interaction of oscillations with thermal normal excitations. The ensemble of such excitations in essence plays a role of the heat bath. In all cases the relatively high temperatures  $T \gg \hbar\omega_0$  have been considered,  $\omega_0$  being the frequency of a parabolic trap. However, the principal question about the origin of the irreversible damping in the oscillating isolated system at  $T = 0$  remains open.

In the present work it is shown that such damping at zero temperature, at least at some definite conditions, really exists. We consider the radial oscillations of the condensate in an elongated trap with the cylindric symmetry, which are induced by the rapid decrease of the frequency  $\omega(t)$  of the transverse isotropic potential from value  $\omega_0$  to  $\omega_1$ . Such statement of the problem has a number of advantages. As is found in [8], there exists an exact scaling solution of the nonlinear Schrödinger equation for a field operator and its quasiclassical analog, Gross-Pitaevskii equation, for an arbitrary behavior  $\omega(t)$  in the case of a gas in the isotropic two-dimensional parabolic potential. It is essential that this solution holds for the quasi-2D case of an elongated trap with the cylindric symmetry. In particular, a ratio between the interaction energy of particles and their kinetic energy remains constant. Thus, if the Thomas-Fermi approximation is applicable in the initial static trap, it remains valid at all stages of the gas evolution.

The scaling solution describes the space-time evolution of the condensate, relying only on the solution in the initial static potential of frequency  $\omega_0$ . After the parameter of the potential takes the value  $\omega_1$ , the condensate sets into the stable state accomplishing with the radial oscillations of frequency  $2\omega_1$ . These oscillations are accompanied by the vibrations of the condensate density and, therefore, by the vibrations of the sound velocity  $c$ .

As will be shown below, for a gas in which the sound velocity oscillates, the phenomenon of parametric resonance appears. The essence of the phenomenon is that

the resonance between the radial condensate vibrations and the longitudinal wave propagating in the direction of the axial  $z$  axis produces an exponential increment for the amplitude of the wave. At zero temperature the triggering amplitudes are formed, in essence, by zero-point oscillations. In the Thomas-Fermi regime, when the chemical potential  $\mu \gg \hbar\omega_0$ , the sound waves of wavelength  $\lambda \approx c/\omega_1$  amplifies. As a result, the dynamic energy of the coherently oscillating condensate reduces, converting into the energy of the longitudinal sound vibrations. For the analysis of the initial stage of generation at  $T = 0$ , it is of interest to find the quantum-mechanical description of the process. Accordingly, in the last section the rate is found for the elementary process of creating a pair of longitudinal phonons in a periodic field of the radially oscillating Bose-condensate.

## THE BOSE-CONDENSATE OSCILLATIONS IN A TRAP

Let us consider a Bose-gas in the elongated cylindric trap with the symmetrical parabolic potential and longitudinal size  $L \gg R_0$ . Neglecting edge effects, we can represent the general equation for the Heisenberg field operator  $\hat{\Psi}(\mathbf{r}, z, t)$  in the cylindric coordinate system as

$$i\hbar \frac{\partial \hat{\Psi}}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla_r^2 - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \frac{1}{2} m \omega^2(t) r^2 \right] \hat{\Psi} + U_0 \hat{\Psi}^+ \hat{\Psi} \hat{\Psi}. \quad (1)$$

Here  $U_0 = 4\pi a \hbar^2 / m$  where  $a$  is the scattering length. The single simplification in (1) is an assumption on the local character of the interparticle interaction.

A unique property of the above equation, containing parabolic 2D potential of frequency  $\omega(t)$  with an arbitrary dependence on the time, is an existence of the scaling transformation. This transformation reduces equation (1) to the form in which the frequency of the parabolic potential is constant and equal to the initial one  $\omega_0$ .

Following [8], let us introduce new scale  $\rho = \mathbf{r}/b(t)$  in the transverse direction and simultaneously define new

time variable  $\tau(t)$ . Representing the field operator as

$$\hat{\Psi}(\mathbf{r}, z, t) = \frac{1}{b(t)} \hat{\chi}(\rho, z, \tau) \exp[i\Phi(r, t)] \quad (2)$$

we insert this expression into (1). In terms of new variables the equation for operator  $\hat{\chi}$  can be reduced to (see Appendix)

$$i\hbar \frac{\partial \hat{\chi}}{\partial \tau} = \left[ -\frac{\hbar^2}{2m} \nabla_\rho^2 + \frac{1}{2} m \omega_0^2 \rho^2 - \mu \right] \hat{\chi} + U_0 \hat{\chi}^\dagger \hat{\chi} \hat{\chi} - b^2(t) \frac{\hbar^2}{2m} \frac{\partial^2 \hat{\chi}}{\partial z^2}, \quad (3)$$

if the phase  $\Phi$  is expressed via the parameters of transformation  $b(t)$  and  $\tau(t)$  by the relation

$$\Phi(r, t) = \frac{mr^2}{2\hbar b} \frac{\partial b}{\partial t} - \frac{\mu \tau(t)}{\hbar}, \quad \mu = \text{const}, \quad (4)$$

and these parameters satisfy the equations

$$\frac{d^2 b}{dt^2} + \omega^2(t) b = \omega_0^2 b^{-3} \quad (5)$$

$$b^2 \frac{d\tau}{dt} = 1. \quad (6)$$

Here  $\omega_0 = \omega(-\infty)$  and Eq. (5) has the following initial conditions  $b(t \rightarrow -\infty) = 1$ . In the case of the problem uniform in the  $z$  direction the equation in variables  $\rho, \tau$  reduces practically to the equations for the static 2D parabolic potential of frequency  $\omega_0$ . Assuming rapid switching at the initial time moment for the trap frequency  $\omega(t)$  from  $\omega_0$  to  $\omega_1$ , we find for the solution of Eq. (5)

$$b^2(t) = \frac{1}{2} (\beta^2 + 1) (1 - g \cos 2\omega_1 t), \quad t > 0 \quad (7)$$

where  $\beta = \omega_0/\omega_1 > 1$  and  $g = (\beta^2 - 1)/(\beta^2 + 1)$ . Thus the parameter  $b(t)$  oscillates at frequency  $2\omega_1$  within the interval from 1 to  $\beta$ .

Substituting this solution into (6) and integrating, we obtain

$$\omega_0 \tau(t) = \omega_1 t + \arctan[\beta \tan(\omega_1 t)] - \arctan[\tan(\omega_1 t)], \quad t > 0 \quad (8)$$

At the stages of small expansion the time  $\tau$  varies at the rate closed to the laboratory one  $t$ . For the stages of the maximum expansion, the time  $\tau$  in the frame moving together with the gas varies slower.

In the general case the nonlinear operator Schrödinger equation (3) describes the non-ideal Bose-gas in the concomitant system of coordinates  $(\rho, \tau)$  in which the gas is in the static potential. A single explicit manifestation of the radial condensate oscillations contains in the last

term on the r.h.s. of (3) describing free motion of the gas along the cylindric axis.

Considering the ground state at  $T = 0$ , we can replace as usually the operator  $\hat{\chi}$  in (3) with the macroscopic wavefunction of condensate  $\chi_0$ . Taking into account that the symmetry of the problem determines an independence of  $\chi_0$  on  $z$ , we arrive at the Gross-Pitaevskii equation, e.g., [10]

$$i\hbar \frac{\partial \chi_0}{\partial \tau} = \left[ -\frac{\hbar^2}{2m} \nabla_\rho^2 + \frac{1}{2} m \omega_0^2 \rho^2 - \mu \right] \chi_0 + U_0 \chi_0^* \chi_0 \chi_0. \quad (9)$$

From the general representation (2) with (4) one can conclude that  $\mu$  is a chemical potential corresponding to the static trap of frequency  $\omega_0$ . In addition, the left-hand side of Eq. (9) vanishes and  $\chi_0$  in variables  $\rho$  and  $\tau$  is a stationary real solution of the equation

$$\left[ -\frac{\hbar^2}{2m} \nabla_\rho^2 + \frac{1}{2} m \omega_0^2 \rho^2 - \mu \right] \chi_0 + U_0 \chi_0^3 = 0 \quad (10)$$

Let us restrict ourselves by the case when the inequality holds for

$$\mu \gg \hbar \omega_0 \quad (11)$$

and therefore the Thomas-Fermi approximation realizes. Then one can neglect the “kinetic energy”, namely, the first term in Eq. (10), and obtain the known solution

$$\chi_0 = \left( \frac{\mu}{U_0} \right)^{1/2} \left( 1 - \frac{\rho^2}{R_0^2} \right)^{1/2}, \quad (12)$$

where

$$R_0 = \sqrt{2\mu/m\omega_0^2} \quad (13)$$

The number of particles and the energy corresponding to this wavefunction are equal to

$$N_0 = \frac{L}{4a} \left( \frac{\mu}{\hbar \omega_0} \right)^2, \quad E_0 = \frac{2}{3} \mu N_0. \quad (14)$$

The energy of the ground state in the trap of frequency  $\omega_1$  equals

$$E_1 = \frac{2}{3} \mu_1 N_0 \quad (15)$$

where

$$\mu_1 = \mu \frac{\omega_1}{\omega_0}. \quad (16)$$

It is worthwhile that solution (12) is valid at the large negative times when  $\omega(t) = \omega_0$  as well as at the large positive times when  $\omega(t) = \omega_1$ . This means that, on one hand, the function  $\chi_0$  is the wavefunction of the ground state of a gas in the trap of frequency  $\omega_0$  and, on the other hand, the same function describes the condensate state at

the large positive times. The state  $\chi_0$  is static only in the variables  $\rho$  and  $\tau$  but in the laboratory frame it oscillates with the frequency  $2\omega_1$  as follows from (7). To describe the space-time evolution of the condensate wavefunction in the variables  $r$  and  $\tau$ , it is sufficient to substitute (12) into the quasiclassical analog of expression (2) and use (4), (7), and (8).

### EXCITATIONS OF THE BOSE-CONDENSATE

With respect to the trap with the parabolic potential of the finite frequency  $\omega_1$ , the oscillating state  $\chi_0$  is an excited state. Naturally, the question arises whether this state is decaying and if it does, what mechanism is. Let us consider the excited states of the system at the background of the coherently oscillating condensate. These oscillations can be found as oscillations of the classical field of the condensate, see, e.g., [11]. For this purpose, let us introduce function  $\chi = \chi_0 + \chi'$  instead of operator  $\hat{\chi}$  in Eq. (3) and linearize it in  $\chi'$

$$i\hbar \frac{\partial \chi'}{\partial \tau} = \left[ -\frac{\hbar^2}{2m} \nabla_\rho^2 + \frac{1}{2} m \omega_0^2 \rho^2 + G - \mu \right] \chi' + G(\chi' + \chi'^*) - \frac{\hbar^2}{2m} b^2(\tau) \frac{\partial^2 \chi'}{\partial z^2}. \quad (17)$$

Here

$$G = U_0 \chi_0^2. \quad (18)$$

We are interested in the long wavelength longitudinal excitations. For  $k_z \equiv k \rightarrow 0$ , as is clear from the physical reasons, the lowest branch of the longitudinal excitations is associated with the uniform shift along the  $z$  axis with the radial distribution of the density determined by the function  $\chi_0$ . Hence this branch is gapless.

For the small finite  $k$ , the transverse distribution of the density in such wave changes weakly. This is clearly seen, e.g, from the results obtained in [12,13] for a static elongated trap. Taking this into account and employing Eq. (10) and definition (18), it is easy to find that the first term on the r.h.s. of (17) is obviously small compared with the term  $G\chi'$ . Then Eq. (17) simplifies significantly

$$i\hbar \frac{\partial \chi'}{\partial \tau} = G(\chi' + \chi'^*) - \frac{\hbar^2}{2m} b^2(\tau) \frac{\partial^2 \chi'}{\partial z^2}, \quad (19)$$

We seek for the solution of the equation for the longitudinal excitation in the form, see, e.g., [11]

$$\chi'(\rho, z, \tau) = u(\rho, \tau) e^{ikz} - v^*(\rho, \tau) e^{-ikz}. \quad (20)$$

Let us introduce the notations

$$f = \frac{1}{\chi_0} (u + v), \quad F = \frac{1}{\chi_0} (u - v)$$

Inserting (20) into Eq. (19) for  $\chi'$  and  $\chi'^*$ , we find

$$i\hbar \frac{\partial F}{\partial \tau} = \frac{b^2 \hbar^2 k^2}{2m} f, \\ i\hbar \frac{\partial f}{\partial \tau} = \left( 2G + \frac{b^2 \hbar^2 k^2}{2m} \right) F.$$

Let us go over in these equations from variable  $\tau$  to the laboratory time  $t$ , using relation (6). Then the equations can be represented as a second-order equation

$$\frac{\partial^2 F}{\partial t^2} + \frac{k^2}{2m} \left( \frac{2G}{b^2(t)} + \frac{\hbar^2 k^2}{2m} \right) F = 0, \quad (21)$$

Let us average Eq. (21) over  $\rho$  and introduce notation  $\langle F \rangle = \bar{F}$ . The average of a product  $GF$ , involving that  $G$  is a positively determined function, can be represented as

$$\langle GF \rangle = \bar{G} \bar{F}$$

As a result, we obtain

$$\frac{\partial^2 \bar{F}}{\partial t^2} + \Omega_k^2(t) \bar{F} = 0, \quad (22)$$

$$\Omega_k^2(t) = \frac{k^2}{2m} \left( \frac{2\bar{G}}{b^2(t)} + \frac{\hbar^2 k^2}{2m} \right). \quad (23)$$

For the case of the static potential  $b = 1$ ,  $\Omega_k$  is nothing else than the frequency of the Bogoliubov spectrum for the longitudinal excitations, determined for some average condensate density. The sound velocity equals

$$\bar{c}_0 = \sqrt{\frac{\bar{G}}{m}} \quad (24)$$

In the problem which we consider, the scaling parameter  $b$  oscillates in time. As follows from (23), this results in the oscillation of the sound velocity. As will be seen below, we are interested in the longitudinal phonons of frequencies  $\Omega_k \simeq \omega_1$ . Owing to inequality (11) this is the purely acoustic region. Accordingly,

$$\Omega_k(t) = c(t) k, \quad c(t) = \frac{\bar{c}}{b(t)}. \quad (25)$$

Using explicit form of the dependence  $b(t)$  (7), we have for the coefficient  $\Omega_k^2(t)$  in Eq. (23)

$$\Omega_k^2(t) = \frac{\omega_k^2}{(1 - g \cos 2\omega_1 t)} \quad (26)$$

where

$$\omega_k^2 = \frac{2(\bar{c}k)^2}{\beta^2 + 1} \quad (27)$$

Thus, Eq. (22) is reduced to the known Hill equation. For the relatively weak variation of the trap frequency when  $g \simeq \beta - 1 \ll 1$ , we arrive at the Mathieu equation

$$\frac{\partial^2 F}{\partial t^2} + \omega_k^2 (1 + g \cos 2\omega_1 t) F = 0. \quad (28)$$

This equation, as well as a more general Hill equation, determines the parametric resonance connecting the coherent transverse condensate oscillations at frequency  $2\omega_1$  with longitudinal phonons with the frequencies close to  $\omega_1$ . From the physical point of view the parametric resonance arises due to periodic variation of the condensate density and, therefore, of the sound velocity (25).

### THE PARAMETRIC RESONANCE

To clarify the conditions for appearing the parametric resonance, we restrict ourselves by the case  $g \ll 1$  and employ the standard algorithm for solving Mathieu equation, see e. g. [9]. We seek for a solution as

$$F = A(t) \cos \omega_1 t + B(t) \sin \omega_1 t. \quad (29)$$

The coefficients  $A(t)$  and  $B(t)$  are assumed to be the slowly varying functions of the time  $t$  and on the substitution of (29) into the Mathieu equation (28) one should retain only the terms zero and linear in  $g$ , neglecting second derivatives of  $A(t)$  and  $B(t)$ . As a result, we arrive at a set of two algebraic linear equations with the constant coefficients. One of the solutions grows exponentially.

$$A(t), B(t) \sim e^{\gamma_k t}, \quad \gamma_k = \frac{1}{2} \sqrt{\left(\frac{1}{2} g \omega_1\right)^2 - s_k^2}, \quad (30)$$

$$s_k = 2(\omega_k - \omega_1),$$

provided that the frequency lies within the narrow band near  $\omega_1$  with the width

$$\delta\omega = t_1^{-1} = \frac{1}{2} g \omega_1. \quad (31)$$

Within this band a growth of the amplitude of the sound waves takes place and therefore the energy converts into the longitudinal modes. As the result the energy of the coherent transverse condensate oscillations decreases. It is necessary to have at least one mode of the longitudinal sound excitations within the energy band of the parametric resonance (31). The spacing between the neighbor modes equals  $\Delta\omega = 2\pi\bar{c}/L$ . Since  $R_0 \simeq 2\bar{c}/\omega_1$ , this condition holds for if there is an inequality

$$L > 2\pi R_0 / g. \quad (32)$$

Within the narrow band around  $\omega_1$  a square of the wavefunction amplitudes (30) increases as  $\exp 2\gamma_k t$ . At  $T = 0$  the initial energy equals the energy of a mode of

zero-point oscillations. Thus the total energy transferred into the longitudinal modes due to parametric resonance equals

$$E(t) \simeq \sum_k \frac{1}{2} \hbar \omega_k (e^{2\gamma_k t} - 1) \theta(\delta\omega - |s_k|). \quad (33)$$

Let us suppose that a sufficient number of modes lies within the band (31) and a sum in (33) can be replaced with an integral

$$E(t) = \frac{gL\hbar\omega_1^2}{16\pi\bar{c}} J(t), \quad (34)$$

$$J(t) = \int_{-1}^1 dx \left( \exp \left[ \frac{t}{t_1} \sqrt{1-x^2} \right] - 1 \right)$$

Let us compare the energy (34) with the energy of the coherent condensate oscillations. For an instantaneous transition of the trap frequency from  $\omega_0$  to  $\omega_1$ , the wavefunction  $\chi_0$  does not change its configuration. Hence the energy of the condensate falls from the initial magnitude  $E_0$  (see (14)) to the magnitude

$$E' = \frac{1}{2} \left( \frac{\omega_1^2}{\omega_0^2} + 1 \right) E_0.$$

The energy of the vibrational motion  $E_c$  which the condensate possesses is equal to a difference between this energy and the energy of the ground condensate state (15) corresponding to frequency  $\omega_1$

$$E_c = E' - E_1 = \frac{1}{3} \mu N_0 g^2 \quad (35)$$

As a result, the damping of the condensate oscillations is characterized by a ratio

$$\frac{E(t)}{E_c} = \frac{3a\omega_0}{4\pi g\bar{c}} \left( \frac{\hbar\omega_0}{\mu} \right)^3 J(t) \simeq \frac{a}{gR_0} \left( \frac{\hbar\omega_0}{\mu} \right)^3 J(t). \quad (36)$$

For the small times  $J(t) \simeq t/t_1$ , and the ratio (36) reduces to

$$\frac{E(t)}{E_c} \simeq \frac{3}{16} \frac{a\omega_0^2}{\bar{c}} \left( \frac{\hbar\omega_0}{\mu} \right)^3 t. \quad (37)$$

For the times  $t \gg t_1$ , the damping grows exponentially

$$\frac{E(t)}{E_c} \simeq \frac{3}{2\pi} \frac{a}{gR_0} \left( \frac{\hbar\omega_0}{\mu} \right)^3 \left( \frac{2\pi t_1}{t} \right)^{1/2} \exp \frac{t}{t_1}. \quad (38)$$

The factor in front of the exponential is much smaller compared with unity for the realistic values of  $g$ . Hence, within the logarithmic accuracy the typical time for decaying the condensate oscillations is equal to

$$t_* = \frac{2}{g\omega_1} \ln \left\{ \frac{gR_0}{a} \left( \frac{\mu}{\hbar\omega_0} \right)^3 \left( \frac{t_*}{2\pi t_1} \right)^{1/2} \right\}. \quad (39)$$

This time is large compared with the period of condensate oscillations not only due to the smallness of  $g$  but also due to the large factor in the argument of logarithm. Thus the damping of the condensate oscillations due to the mechanism considered is a rather slow process though appearing inevitably at  $T = 0$  under conditions concerned.

The parametric resonance leads to reducing the number of particles in the condensate. In principle, this could require to solve a self-consistent problem. However, from the general considerations it is clear that the escape of particles from the condensate can be neglected if the vibrational energy per particle  $g^2\mu$  (see (35)) is small compared with the temperature of Bose-condensation  $T_c$ .

### THE QUANTUM CALCULATION OF THE PHONON GENERATION AT THE INITIAL STAGE

The classical parametric resonance develops from the moment  $t = 0$  only in the case when nonzero amplitude of the resonant modes within the interval  $\delta\omega$  (31) exists from the very beginning. At the finite temperature these are amplitudes of thermal oscillations. At  $T = 0$  the initial amplitudes should be connected with the zero-point oscillations of the longitudinal modes. This is explicitly taken into account in the preceding section. It is of interest to trace the initial stage of the phonon generation, remaining within the framework of the purely quantum mechanical consideration. For this purpose, we should return to the operator description of excitations in the system.

The general equation for operator  $\hat{\chi}'$  has the form analogous to that of Eq. (17)

$$i\hbar \frac{\partial \hat{\chi}'}{\partial \tau} = h_0 \hat{\chi}' + G (\hat{\chi}' + \hat{\chi}'^+) - \frac{\hbar^2}{2m} b^2(\tau) \frac{\partial^2 \hat{\chi}'}{\partial z^2}, \quad (40)$$

$$h_0 = -\frac{\hbar^2}{2m} \nabla_\rho^2 + \frac{1}{2} m \omega_0^2 \rho^2 + G - \mu \quad (41)$$

We go over from variable  $\tau$  to  $t$  (see (6)) and single out the term associated with the coherent condensate oscillations in the explicit form

$$i\hbar \frac{\partial \hat{\chi}'}{\partial t} = h \hat{\chi}' + [h_0 \hat{\chi}' + G (\hat{\chi}' + \hat{\chi}'^+)] (b^{-2}(\tau) - 1), \quad (42)$$

$$h = h_0 - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2}$$

Assuming that  $(b^{-2}(\tau) - 1) \ll 1$ , we treat the last term on the r.h.s. as a perturbation. From the standard commutation rule for the field operators  $\hat{\Psi}(\mathbf{r}, z, t)$  and representation (2) it directly follows the commuta-

tion rule for operators  $\hat{\chi}'$

$$[\hat{\chi}'(\rho, z, t), \hat{\chi}'^+(\rho', z', t)] = b^2 \delta(\mathbf{r} - \mathbf{r}') \delta(z - z') = \delta(\rho - \rho') \delta(z - z')$$

Using this relation, one can readily write the perturbation hamiltonian governing the behavior of the last term on the r.h.s. of Eq. (42)

$$H' = (b^{-2}(\tau) - 1) \times \int d^2 \rho dz \left[ \hat{\chi}'^+ (h_0 + G) \hat{\chi}' + \frac{1}{2} G (\hat{\chi}' \hat{\chi}' + \hat{\chi}'^+ \hat{\chi}'^+) \right] \quad (43)$$

Consider at first the excitations in the static case when  $b = 1$ . In the secondary quantization the operator  $\hat{\chi}'$  can be written using the eigenfunctions of hamiltonian  $h$  as

$$\hat{\chi}' = \sum_{nk} \hat{a}_{nk} \chi_{nk}, \quad \chi_{nk} = \frac{e^{ikz}}{\sqrt{L}} \varphi_n(\rho), \quad (44)$$

$$h_0 \varphi_n(\rho) = E_n \varphi_n(\rho)$$

Inserting (44) into Eq. (42), multiplying both sides of the equation by  $\chi_{nk}^*$ , and integrating over  $\rho$  and  $z$ , we find

$$i\hbar \frac{d\hat{a}_{nk}}{dt} = \left( E_n + \frac{\hbar^2 k^2}{2m} \right) \hat{a}_{nk} + G_{nn} (\hat{a}_{nk} + \hat{a}_{nk}^+) \quad (45)$$

where

$$G_{nn} = \int d^2 \rho \varphi_n^2(\rho) G(\rho)$$

Equation (45) is approximate since we have omitted the terms nondiagonal in  $n$  and proportional to  $G_{nn'}$ . The involvement of these terms leads to the quantitative corrections alone. Employing the standard Bogoliubov transformation

$$\hat{a}_{nk} = u_{nk} \hat{b}_{nk} - v_{nk} \hat{b}_{n-k}^+, \quad (46)$$

and obvious condition

$$i\hbar \frac{d\hat{b}_{nk}}{dt} = \varepsilon_{nk} \hat{b}_{nk}$$

for independent collective excitations, we find the spectrum of excitations and the values of coefficients  $u_{nk}, v_{nk}$ . For the reasons mentioned in the preceding section, we consider only the phonon generation corresponding only to the lowest branch of excitations  $n = 0$ . For  $\varepsilon_{nk}$  in this case, we find the expression coinciding with (23) at  $b = 1$  if  $G_{00}$  is understood as  $\bar{G}$ . Now let us find the phonon generation in first approximation in  $H'$  (43). For this purpose, using representation (44), we perform transformation (46), considering only the terms with  $n=0$ . Since

we consider the case  $T=0$ , the corresponding generation proves to be associated only with the terms having a product  $\hat{b}_{0k}^+ \hat{b}_{0-k}^+$  in the hamiltonian, i.e., with the production of two phonons with the opposite momenta. Correspondingly we find directly for the transition amplitudes

$$A_{k,-k} = \frac{1}{2} g \bar{G} \cos(2\omega_1 t) (u_{0k} - v_{0k})^2 \times \langle k, -k | \hat{b}_{0k}^+ \hat{b}_{0-k}^+ | 0 \rangle \quad (47)$$

Thus, at the initial stage of relaxation the probability of generating phonon pairs per unit time is equal to

$$W_{k,-k} = \frac{\pi}{8\hbar} [g \bar{G} (u_{0k} - v_{0k})^2]^2 \delta(2\hbar\omega_1 - 2\varepsilon_{0k}). \quad (48)$$

Using the known relations for the coefficients of the Bogoliubov transformation, e.g., [10], we have for the region of sound excitations (see (27))

$$(u_{0k} - v_{0k})^2 \simeq \frac{\varepsilon_{0k}}{\mu}, \quad \varepsilon_{0k} = \hbar\omega_k$$

Hence we readily find the power of the energy lost by the oscillating condensate

$$\dot{E} = \frac{L}{2\pi\hbar c} \int d\varepsilon 2\varepsilon W_{k,-k} = \frac{L\hbar}{64c} g^2 \omega_1^3. \quad (49)$$

This result coincides with that (37) (taking into account (35) and (14)) obtained in the preceding section for the parametric resonance at the initial times  $t \ll t_1$ . The quantum mechanical calculation confirms our notion on generating the longitudinal waves at zero temperature as an enhancement of zero-point oscillations due to parametric resonance.

## THE PARAMETRIC RESONANCE AT THE MULTIPLE FREQUENCIES

As is known, the Mathieu equation (28) results in the parametric resonance at the frequencies close to the multiple frequencies  $p\omega_1$ . However, considering the phonon generation at these frequencies, we must return to the general equation (22) even in the case  $g \ll 1$ . The analysis of the equation shows that the parametric resonance is absent for the even multiple frequencies. The phonon generation can occur only near the multiple frequencies  $p\omega_1$  with odd  $p$ . As  $p$  grows, the growth decrement  $E(t)$  and the width of the resonant interval decrease strongly at any value of  $g$ . This can easily be traced, analyzing the nearest multiple resonance of  $p = 3$ . Such unusual result is due to specific behavior of the oscillating coefficient in Eq. (22) varying periodically in time.

## CONCLUSIVE REMARKS

Thus, analyzing the coherent radial oscillations of the condensate in a cylindric trap, we provide the evidence

for the existence of damping of these oscillations at zero temperature. The damping is a consequence of the parametric resonance caused by the sound velocity oscillations. As a result the energy transfer from radial condensate oscillations to longitudinal phonon modes is realized.

At the finite temperature  $\hbar\omega_1 < T \ll \mu$  the same resonance amplification takes place but now the initial number of phonons with the energy  $\hbar\omega_1$  equals  $T/\hbar\omega_1$  and the initial energy of the resonance modes equals  $\hbar\omega_1 \cdot (T/\hbar\omega_1)$ . For this case, in Eq. (33) the quantity  $\hbar\omega_1$  should be replaced with  $T$ . Such replacement increases insignificantly the argument in logarithm (39).

Let us estimate the characteristic damping time  $t_*$  for quantum gases *Rb* and *Na*. We consider an elongated cylindric trap with  $\nu_0 = 400\text{Hz}$  (*Ru*) or  $\nu_0 = 10^3\text{Hz}$  (*Na*),  $L = 2 \cdot 10^5$  and assume that the dimensionless parameter  $g \approx 0.15$ . Then from Eq. (14) we find  $\mu \approx 90nK$  for the *Rb* gas. Accordingly, the Thomas-Fermi radius equals  $R \approx 1.7 \cdot 10^{-4}\text{cm}$ . For the parameter  $t_1$  (31), taking into account the relation  $\omega_1 = (1 - g) \cdot \omega_0$ , we have  $t_1 \approx 6.2 \cdot 10^{-3}\text{s}$ . The direct calculation estimates the logarithmic factor in Eq. (39) as  $\sim 10$ . Thus, evaluating the damping time, we find  $t_* \leq 0.1\text{s}$ . Since it is smaller than the ordinary lifetime of the system, this time is realistic for revealing the damping.

The similar results can be obtained for sodium. In this case the estimates give  $\mu \approx 140nK$ ,  $R \approx 1.6 \cdot 10^{-4}\text{cm}$ , and  $t_1 \approx 2.5 \cdot 10^{-3}\text{s}$ . As a result,  $t_* \approx 0.03\text{s}$ . It is important that in both cases the condition (32) proves to be fulfilled. At the same time  $g\mu \ll T_c$ , and this means that the depletion of the condensate is quite small after the complete damping.

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